SOLUTION OF INTERNAL PROBLEMS OF AERODYNAMICS UNDER TRANSITIONAL CONDITIONS USING A MODEL KINETIC EQUATION

A. I. Ivanovskii and S. Kh. Rozenfel'd

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A method is developed based on the use of a model kinetic equation with a shock frequency $\omega = v/l$ (l is the mean length of the free flight path; v is the modulus of the molecular velocity). The method is tested on several classical problems.

1. Description of Method. We shall use as basis the model kinetic equation

$$\mathbf{v}\frac{\partial f}{\partial \mathbf{r}} = \frac{1}{\tau(v)}(f_0 - f) \tag{1.1}$$

Here v is the molecular velocity; $\tau^{-1}(v)$ is the shock frequency, depending on the velocity; f_0 is the local-equilibrium distribution function. In the consideration of the internal flows of a rarefied gas, in the majority of cases we can limit ourselves within the framework of a linear approximation (i.e., with small Mach numbers and small temperature gradients).

Therefore, we take

$$f_0(\mathbf{v}, \mathbf{r}) = N(2\pi m\theta)^{-3/2}\exp\left(-\frac{p^2}{2m\theta}\right)\left(1 + \frac{\mathbf{pu}}{\theta}\right)$$

We divide all the particles into two sorts: primary particles which have just flown away from the wall and have not undergone even one collision; secondary particles, which have undergone at least one collision. We write the kinetic equations for each sort separately:

$$\mathbf{v}\frac{\partial f_2}{\partial \mathbf{r}} = \frac{1}{\tau}(f_{02} - f_2) + \frac{1}{\tau}f_{01}, \quad \mathbf{v}\frac{\partial f_1}{\partial \mathbf{r}} = -\frac{1}{\tau}f_1 \tag{1.2}$$

Here f_1 and f_2 are respectively the distribution functions of the primary and secondary particles; f_{01} and f_{02} are local-equilibrium functions normalized respectively for the densities of the number of primary and secondary particles.

The necessity for such a separation is a result of the following: with collisions between secondary particles the momentum and the energy of any given element of the volume are retained; with collisions between primary and secondary particles, in each element of the volume, there appears a momentum and an energy brought in by the particle from that point of the surface from which it was emitted. Therefore, at each point of its volume a gas consisting of secondary particles has sources of energy and momentum formed by the flows of primary particles at the given point.

Let us make the form of the function $\tau(v)$ definite. We take $\tau(v) = l/v$. Here l is the mean length of the free-flight path; v is the molecular velocity. (Various means for selecting $\tau(v)$ are discussed in [1].)

We write the laws of conservation for the secondary particles:

$$\int d\mathbf{p} f_2 v = \int d\mathbf{p} f_{02} v = \frac{2}{\sqrt{\pi}} N_2 v_0 \tag{1.3}$$

$$\int d\mathbf{p} f_2 \mathbf{v} v = \int d\mathbf{p} f_{02}\mathbf{v} v = \frac{8}{3\sqrt{\pi}} N_2 v_0 \mathbf{u}, \quad v_0 = \left(\frac{2\theta}{m}\right)^{1/2} \tag{1.4}$$

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$$\int d\mathbf{p} f_2 v v^2 = \int d\mathbf{p} f_{02} v v^2 = \frac{8}{\sqrt{\pi}} N_2 v_0 \frac{v_0^2}{2} \tag{1.5}$$

With the given selection of τ (v), Eqs. (1.2) can be rewritten as

$$f_1(\varkappa v, \mathbf{r}_s + \varkappa s) = f_1(\varkappa v, \mathbf{r}_s) e^{-s/l} \tag{1.6}$$

$$f_2(\varkappa v, \mathbf{r}_s + \varkappa s) = f_2(\varkappa v, \mathbf{r}_s) f^{-s/l} + \int_0^s ds' e^{-(s'-s)/l} f_0(\varkappa v, \mathbf{r}_s + \varkappa s') \tag{1.7}$$

Here $\varkappa = \mathbf{v}/v$, $f_0 = f_{01} + f_{02}$; $\mathbf{r}_s$ is the radius vector of a point on the surface; f_s is the distribution function of the particles flying away from the surface; we assume that this distribution is locally Maxwellian.

We now fix the point $\mathbf{r} = \mathbf{r}_s + \varkappa s$ and substitute expression (1.7), consecutively into the equations of conservation (1.3)-(1.5):

$$\frac{2}{\sqrt{\pi}} N_2 v_0 = \frac{1}{l} \int d\mathbf{r}' \frac{e^{-R/l}}{4\pi R^2} \left\{ \frac{2}{\sqrt{\pi}} N v_0 + 3\varkappa \mathbf{Q} \right\} \tag{1.8}$$

$$\frac{8}{3\sqrt{\pi}} \mathbf{Q}_2 v_0 = \frac{1}{l} \int d\mathbf{r}' \frac{e^{-R/l}}{4\pi R^2} \left\{ \frac{8}{3\sqrt{\pi}} v_0 \varkappa (\varkappa \mathbf{Q}) + 3\varkappa \frac{N v_0^2}{2} \right\} \tag{1.9}$$

$$\frac{8}{\sqrt{\pi}} N_2 v_0 \frac{v_0^2}{2} = \frac{1}{l} \int d\mathbf{r}' \frac{e^{-R/l}}{4\pi R^2} \left\{ \frac{8}{\sqrt{\pi}} N \frac{v_0^2}{2} v_0 + \frac{15}{4} N v_0^2 (\varkappa \mathbf{Q}) \right\} \tag{1.10}$$

Here

$$\varkappa = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}, \qquad R = |\mathbf{r} - \mathbf{r}'|, \qquad \mathbf{Q} = N\mathbf{u}$$

Equations (1.8)-(1.10) express the density, the flows, and the pressure of the secondary particles in terms of the total values of these quantities over the whole volume. We now find the contribution of the primary particles. Integrating Eq. (1.6) over the space of the momenta at the point $\mathbf{r} = \mathbf{r}_s + \varkappa s$, we find

$$N_1(\mathbf{r}) = -\oiint d\mathbf{S}' \varkappa \frac{e^{-R/l}}{4\pi R^2} \left\{ N_s + \frac{4}{\sqrt{\pi}} N_s \frac{\mathbf{u}_s \varkappa}{v_{0s}} \right\} \tag{1.11}$$

$$Q_1(\mathbf{r}) = -\oiint d\mathbf{S}' \varkappa \frac{e^{-R/l}}{4\pi R^2} \left\{ \frac{2}{\sqrt{\pi}} v_0 N_s \varkappa + 3\varkappa (N_s \mathbf{u}_s \varkappa) \right\} \tag{1.12}$$

$$N_1 \frac{v_0^2}{2} = -\oiint d\mathbf{S}' \varkappa \frac{e^{-R/l}}{4\pi R^2} \left\{ N_s \frac{v_{0s}^2}{2} + \frac{8}{3\sqrt{\pi}} v_{0s} (N_s \mathbf{u}_s \varkappa) \right\} \tag{1.13}$$

Here u_s is the velocity of the wall; $v_{0s} = (2\theta_s/m)^{1/2}$; θ_s is the temperature of the wall.

The total density, flows, and pressure are expressed by the formulas

$$N = \frac{1}{l v_0} \int d\mathbf{r}' \frac{e^{-R/l}}{4\pi R^2} \left\{ N v_0 + \frac{3\sqrt{\pi}}{2} \varkappa \mathbf{Q} \right\} - \oiint d\mathbf{S}' \varkappa \frac{e^{-R/l}}{4\pi R^2} \left\{ N_s + \frac{\varkappa \mathbf{u}_s}{v_{0s}} N_s \right\} \tag{1.14}$$

$$\mathbf{Q} = \frac{1}{l v_0} \int d\mathbf{r}' \frac{e^{-R/l}}{4\pi R^2} \left\{ 3\varkappa v_0 (\varkappa \mathbf{Q}) + \frac{9\sqrt{\pi}}{8} \varkappa \frac{N v_0^2}{2} \right\} - \oiint d\mathbf{S}' \varkappa \frac{e^{-R/l}}{4\pi R^2} \left\{ \frac{2}{\sqrt{\pi}} N_s v_{0s} \varkappa + 3\varkappa (\varkappa N_s \mathbf{u}_s) \right\} \tag{1.15}$$

$$N \frac{v_0^2}{2} = \frac{1}{l v_0} \int d\mathbf{r}' \frac{e^{-R/l}}{4\pi R^2} \left\{ N \frac{v_0^2}{2} v_0 + \frac{15\sqrt{\pi}}{32} v_0^2 \varkappa \mathbf{Q} \right\} - \oiint d\mathbf{S}' \varkappa \frac{e^{-R/l}}{4\pi R^2} \left\{ N_s \frac{v_{0s}^2}{2} + \frac{8}{3\sqrt{\pi}} v_{0s} (N_s \mathbf{u}_s \varkappa) \right\} \tag{1.16}$$

The parameters N_s and Q_s are determined from the condition of nonflow

$$\mathbf{Q}(\mathbf{r}_s) \mathbf{n}(\mathbf{r}_s) = 0$$

The system of equations (1.14)-(1.17) is a closed system of integral equations, sufficient in principal for the solution of any given problem involving the flows of a rarefied gas.

We consider below a number of problems whose solutions are well known; we shall use these as examples to demonstrate the correctness and the very high efficiency of the method developed here.

2. Couette Flow. Let there be two infinite flat plates, moving parallel one to the other at velocities of $\pm u_s$. The distance between the plates is equal to $2a$. There is sought the flow density of the particles along the axis of the plates (Fig. 1).

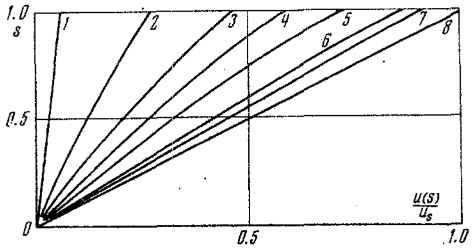

Fig. 1

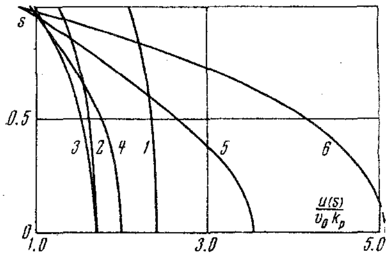

Fig. 2

In this case also, the integral equation (1.15) for the flow can be transformed to the following form:

$$Q(s) = \int_{-1}^{+1} K(s, s') Q(s') ds' + \frac{3}{2} N_s u_s \int_1^\infty \frac{dt}{t^2}\left(1 - \frac{1}{t^2}\right) e^{-\lambda t} \operatorname{sh} \lambda t s$$

where

$$K(s, s') = \frac{3}{2} \frac{\lambda}{2} \int_1^\infty \frac{dt}{t}\left(1 - \frac{1}{t^2}\right) e^{-\lambda t |s-s'|} \tag{2.1}$$

Here $s = z/a$. The parameter $\lambda = a/l = \mathrm{Kn}^{-1}$ characterizes the degree of rarefaction. The integral equation (2.1) was solved numerically over a wide range of values of λ from 10^{-3} to 5. Typical flow profiles are shown in Fig. 1, where the values of λ = 0.01, 0.1, 0.3, 0.5, 1.0, 3.0, 5.0, ∞ correspond to curves 1-8.

The limiting cases can be investigated analytically. Thus, with $\lambda \to 0$, we have $Q(s) \equiv 0$. With $\lambda \to \infty$, we use the asymptotic formula

$$\lim_{\lambda \to \infty} \int_{-1}^{+1} f(s') ds' \int_1^\infty \frac{dt}{t^n} e^{-\lambda t |s-s'|} = \frac{2}{\lambda}\left\{\frac{1}{n} f(s) + \frac{1}{n+2} \frac{1}{\lambda^2} \frac{\partial^2 f}{\partial s^2}\right\} \tag{2.2}$$

Applying (2.2) to Eq. (2.1), with $\lambda \to \infty$, we obtain $\partial^2 u/\partial s^2 = 0$, with slipping conditions at the boundary:

$$u(1) = u_s - \frac{3}{8} l \frac{\partial u}{\partial z} \tag{2.3}$$

The solutions of Eq. (2.3) have linear profiles passing through the origin of coordinates. A comparison between conditions (2.3) and Fig. 1 shows that already at $\lambda = 5$ the solution of integral equation (2.1) coincides with its continuous asymptotic curve.

The results set forth here are in good agreement with known data (see, for example, [2]).

3. Poiseuille Flow. We consider an isothermal flow, arising under the effect of a constant pressure gradient applied along the X axis. In this case, Eq. (1.15) is transformed to the form

$$Q^*(s) = \int_{-1}^{+1} K(s, s') Q^*(s') ds' - \frac{9\sqrt{\pi}}{8}\left\{\frac{k_p}{\lambda}\left(\frac{1}{3} - \int_1^\infty \frac{dt}{t^4} e^{-\lambda t} \operatorname{ch} \lambda t s\right) - \frac{16}{9\pi} \frac{k_p}{\lambda} \int_1^\infty \frac{dt}{t^2}\left(1 - \frac{3}{t^2}\right) e^{-\lambda t} \operatorname{ch} \lambda t s\right\} \tag{3.1}$$

Here

$$Q^* = \frac{2u}{v_0}, \qquad k_P = \frac{a}{P} \frac{\partial P}{\partial x}$$

This integral equation was solved numerically from $\lambda = 10^{-3}$ to $\lambda = 5.0$. Figure 2 shows characteristic flow profiles for values of λ = 0.01, 0.1, 0.5, 1.0, 3.0, 5.0 (curves 1-6, respectively). A comparison with existing data [2] shows almost total agreement. With $\lambda \to \infty$, from Eq. (3.1) there can be obtained the well-known Poiseuille equation

$$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{\eta} \frac{\partial P}{\partial x} \tag{3.2}$$

with slipping conditions at the boundary:

$$u(1) = \frac{3\sqrt{\pi}}{4} v_0 \frac{k_p}{\lambda} - \frac{3}{8} \frac{1}{\lambda} \frac{\partial u}{\partial s}(1) \tag{3.3}$$

Under these circumstances the viscosity coefficient is equal to

$$\eta = \frac{8}{15\sqrt{\pi}} \rho v_0 l \tag{3.4}$$

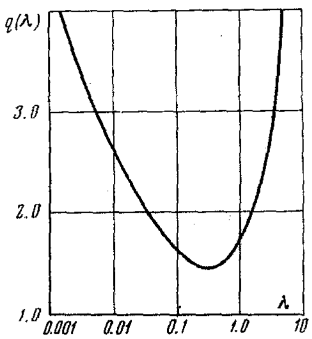

Fig. 3

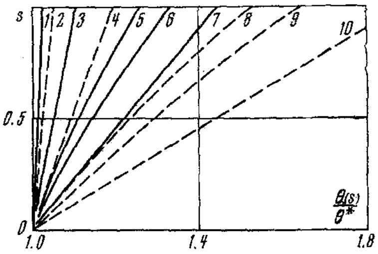

Fig. 4

An important special characteristic of plane Poiseuille flow is the existence of a minimum of the mass flow rate at certain values of λ (the Knudsen paradox). Figure 3 shows the mass flow rate curve obtained by the present authors. At $\lambda \to 0$ it has a logarithmic singularity, at $\lambda \to \infty$ it rises linearly, and it attains a minimum at $\lambda = 0.36$–0.4. In [3] a value of $\lambda = 0.42$–0.55 is given. Close values were obtained in recently conducted experiments [4].

4. Heat Transfer between Flat Plates. We shall first calculate the temperature profile between infinite flat plates, of which the upper has a temperature of θ_+, and the lower a temperature of θ_-. Integral equation (1.16) is transformed to the form

$$\theta(s) = \int_{-1}^{+1} K(s, s')\,\theta(s')\,ds' + \int_{1}^{\infty} \frac{dt}{t^2} e^{-\lambda t} \left\{ \frac{\theta_+ + \theta_-}{2} \operatorname{ch} \lambda t s + \frac{\theta_+ - \theta_-}{2} \operatorname{sh} \lambda t s \right\}$$

$$K(s, s') = \frac{\lambda}{2} \int_{1}^{\infty} \frac{dt}{t} e^{-\lambda t |s-s'|}. \tag{4.1}$$

The results of a numerical calculation of $\theta(s)$ are given in Fig. 4. Curves 1–10 correspond to the following pairs of values: λ, δ = 0.01, 0.5; 0.01, 1.0; 0.1, 0.5; 0.1, 1.0; 0.5, 0.5; 1.0, 0.5; 5.0, 0.5; 0.5, 1.0; 1.0, 1.0; 5.0, 1.0; $\delta = (\theta_+ - \theta_-)/(Q_+ + Q_-)$.

The heat flux is expressed in terms of the temperature in the following manner:

$$J(s) = -\frac{4}{(2\pi m)^{1/2}} \int_{-1}^{+1} ds' \frac{\partial}{\partial s'} \theta^{3/2}(s') \int_{1}^{\infty} \frac{dt}{t^3} e^{-\lambda t |s-s'|}. \tag{4.2}$$

At the free-molecular limit this expression goes over into

$$J(s) = -\frac{2}{(2\pi m)^{1/2}} (\theta_+^{3/2} - \theta_-^{3/2}). \tag{4.3}$$

At the continuous limit the temperature profiles are linear with a temperature discontinuity at the walls:

$$\theta(1) = \theta_+ - \frac{1}{2} l \frac{\partial \theta}{\partial z}(1). \tag{4.4}$$

At $\lambda \to \infty$, formula (4.2) goes over into the thermal-conductivity equation with the thermal-conductivity coefficient

$$k_T = \frac{2}{\sqrt{\pi}} \rho v_0 l. \tag{4.5}$$

Disposing of explicit expressions for the viscosity and thermal-conductivity coefficients, we can find the Prandtl number

$$\Pr = \frac{c_p \eta}{k_T} = \frac{2}{3}.$$

This value coincides with the Prandtl number for the Boltzmann equation.

Thus, the proposed method ensures a correct limiting transition both to free-molecular and to continuous flows. At the same time, without imposing any sort of restrictions on the geometry of the region and the dimensionality of the problem, it permits a very economical description of flows with arbitrary Knudsen numbers.

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